Math 245C Lecture 19 Notes

Daniel Raban

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1 The Fourier Transform and Derivatives

1.1 How the Fourier transform interacts with derivatives

Theorem 1.1. *Let $f \in L^1$. Then the following hold.*

1. *If $x^\alpha f \in L^1$ for all $|\alpha| \leq k$, then*

$$\partial^\alpha \widehat{f}(\xi) = \widehat{(-2\pi i x)^\alpha f}(\xi).$$

2. *If $f \in C^k$, $\partial^\alpha f \in L^1 \cap C_0$ for $|\alpha| \leq k-1$, and $\partial^\alpha f \in L^1$ for $|\alpha| = k$, then*

$$\widehat{\partial^\alpha f}(\xi) = (2\pi i \xi)^\alpha \widehat{f}(\xi).$$

Proof. For the first statement, we will show the proof of $|\alpha| = 1$. The rest will follow by induction on $|\alpha|$. Let $\xi \in \mathbb{R}^n$. Then

$$\widehat{f}(\xi + h) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} e^{-2\pi i h \cdot x} f(x)\, dx.$$

If $h = te_j$, then

$$\frac{\widehat{f}(\xi + te_j) - \widehat{f}(\xi)}{t} = \int_{\mathbb{R}^n} e^{2\pi i \xi \cdot x} \frac{e^{-2\pi i t x_j} - 1}{t}\, dx.$$

Using a first order Taylor expansion of the exponential, we get $|e^{-2\pi i x_j t}|/|t| \leq 2\pi |x_j|$. So, using the dominated convergence theorem, since $2\pi |x_j||f(x)| \in L^1$,

$$\partial^\alpha \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{2\pi i \xi \cdot x}(-2\pi i x_j)\, dx = \widehat{(-2\pi i x) f}(\xi).$$

For the second statement, we want to understand why we need $f \in C_0 \cap L^1$ and $\partial_{x_j} f \in L^1$ to have $\widehat{\partial_{x_j} f}(\xi) = (2\pi i \xi_j)\widehat{f}(\xi)$. Assume $k = 1$. Then

$$\widehat{\partial_{x_j} f}(\xi) = \int_{\mathbb{R}^n} \partial_{x_j} f(x) e^{-2\pi i \xi \cdot x}\, dx$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \partial_{x_j} f(x) e^{-2\pi i \sum_{k\neq j} \xi_k x_k} e^{-2\pi i x_j \xi_j}\, dx_j\, dx_1 \cdots dx_{j-1}\, dx_j \cdots dx_n \\
&= \int_{\mathbb{R}^{n-1}} e^{-2\pi i \tilde{\xi \cdot x}^{-j}} \left[- \int_{\mathbb{R}} f(x) \frac{\partial}{\partial_{x_j}} \left(e^{2\pi i \xi_j x_j} \right) + \left[f(x) e^{-2\pi i x_j \xi_j} \right]_{-\infty}^{\infty} \right] \tilde{dx}^{-j} \\
&= \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} f(x)(-2\pi i \xi_j)\, dx \\
&- 2\pi i \xi_j \widehat{f}(\xi)
\end{aligned}
$$

To prove that $\widehat{f} \in C_0$, it suffices to find $(g_k)_k \subseteq C_0$ such that $\lim_k \|\widehat{f} - g_k\|_u = 0$. Let $(f_k)_k \subseteq C_0^\infty(\mathbb{R}^n)$ be such that $\|f - f_k\|_1 \leq 1/k$. We have

$$
\|\widehat{f} - \widehat{f}_j\|_u \leq \|f - f_k\|_1 \leq \frac{1}{k}.
$$

But $(2\pi i \xi_j)\widehat{f_k} = \widehat{\partial_{x_j} f_k}$. Thus,

$$
2\pi \|\xi_j \widehat{f_k}\|_u \leq \|\partial_{x_j} f_k\|_1 < \infty.
$$

This means that $|\xi||\widehat{f_k}|$ is bounded, and so $\widehat{f_k} \in C_0$. □

1.2 The Fourier transform on the Schwarz space

Corollary 1.1. *$\mathcal{F}$ maps $\mathcal{S}$ into $\mathcal{S}$ continuously.*

Proof. Let $f \in \mathcal{S}$. We are to control the uniform norm of $x^a \partial^b \widehat{f}$ for all multi-indices $a, b \in \mathbb{N}^n$ using a finite number of expressions $\|f\|_{(N_i, \alpha_i)}$. Since $x^a \partial^b \widehat{f}$ is a finite linear combination of terms of the form $\partial^\beta (x^\alpha \widehat{f})$, it suffices to control the latter expressions. Note that

$$
\partial^\beta (x^\alpha \widehat{f}) = \frac{\partial^\beta \left((2\pi i x)^\alpha \widehat{f} \right)}{(2\pi i)^\alpha} = \frac{\partial^\beta (\widehat{\partial^\alpha f})}{(2\pi i)^\alpha} = \frac{1}{(2\pi i)^\alpha} (-2\widehat{\pi i x)^\beta \partial^\alpha} f.
$$

Thus,

$$
\|\partial^\beta (x^\alpha \widehat{f})\|_u \leq |2\pi|^{\beta - \alpha} \|x^\beta \partial^\alpha f\|_1.
$$

The right hand side is

$$
\begin{aligned}
|2\pi|^{\beta-\alpha} & \int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{n+1}} (1+|x|)^{n+1} |x^\beta \partial^\alpha f|\, dx \\
&\leq |2\pi|^{\beta-\alpha} \|(1+|x|)^{n+1+|\beta|} \partial^\alpha f\|_u \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{1+|x|)^{n+1}}\, dx \\
&= |2\pi|^{\beta-\alpha} \|f\|_{(2+1+|\beta|, \alpha)} C_n,
\end{aligned}
$$

where C_n is a constant. □

Remark 1.1. Given $a > 0$ and an integer $n \geq 1$, we define

$$f_a^n(x) = e^{-\pi|x|^2 a}.$$

Note that $f_a^n \in \mathcal{S}$, and

$$f_a^n(x) = \prod_{j=1}^{n} f_a^1(x_j).$$

Hence,

$$\begin{aligned}\widehat{f_a^n}(\xi) &= \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} \prod_{j=1}^{n} f_a^1(x_j)\, dx \\ &= \prod_{j=1}^{n} \int_{\mathbb{R}^n} e^{2\pi i \xi_j x_j} f_a^1(x_j)\, dx_j \\ &= \prod_{j=1}^{n} \widehat{f_a^1}(\xi_j).\end{aligned}$$